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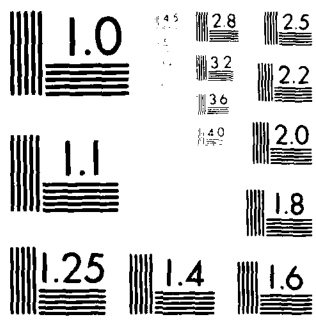

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METRIC 2: THE MATHEMATICAL THEORY OF A SPARING MODEL FOR REPAIRABLES

by
P.A. Vincent

ORAE MEMORANDUM NO. M102

ORAE

OPERATIONAL RESEARCH AND ANALYSIS ESTABLISHMENT
DEPARTMENT OF NATIONAL DEFENCE
OTTAWA, CANADA
MARCH 1980

DEPARTMENT OF NATIONAL DEFENCE

CANADA

OPERATIONAL RESEARCH AND ANALYSIS ESTABLISHMENT

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ORAE ~~MEMORANDUM NO~~ M102

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RESUME

Ce rapport a pour but de décrire METRIC 2, un modèle mathématique viser à la détermination des pièces de rechange. Après une discussion de sa structure générale, ses moyens probabilistes et d'optimisation sont sondés. Enfin, la séquence d'étapes à la solution d'un problème donné est mise en évidence. Cette séquence en effet est celle suivie par le programme informatique utilisé dans la procédure d'approvisionnement initiale d'un équipement majeur (Logistics Management Instruction 1630).

ABSTRACT

This report describes METRIC 2, a mathematical sparing model for repairables. After discussing its general structure, the probability and optimization techniques are explored. Finally the sequence of steps required to solve a given problem is discussed. These are basically the steps used in the FORTRAN program of the model currently used in initial provisioning for capital equipment in accordance with the procedures of CF Logistics Management Instruction 1630.

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FOREWORD

The objective of this memorandum is to centralize and to better explain all the mathematical theory behind METRIC 2, a complex initial provisioning model designed to spare repairables for weapon systems upon procurement. It has been found in the past that analysts taking over responsibility for the model have to locate and draw together several unrelated reports in order to understand the mechanics of the model to a satisfactory level. Often these reports are papers whose abridged contents are aimed at experts in the field. Thus this work will provide a fundamental contribution to the continuity of the support that D Log A must provide for METRIC 2.

As a secondary objective, the consolidation of this material into one document provides an important first step in a project whose mandate is to modify METRIC 2. This work will supply a solid foundation from which future changes to the model, consistent with its current functioning, can be made.

The documentation of the mathematical theory in this memorandum is intended to be self-contained. The four principal parts (the general description, the probability, the optimization and the computations) were designed to give a good working knowledge of the model's theory without proofs or discussions of the more involved aspects. Annexes A through D describe the more complex mathematical tools or proofs used in the main text. Those whose needs or interests require more depth are referred to the respective section. A first course in calculus and probability and statistics is all that is required as background.

METRIC 2: THE MATHEMATICAL THEORY OF A SPARING MODEL FOR REPAIRABLES

I - GENERAL DESCRIPTION OF THE MODEL

BACKGROUND AND OBJECTIVES

1. Inventory control generally deals with two questions: when to reorder and how much to reorder. However, there is a significant class of items where demand is so low or cost so high that reordering immediately upon demand is the best possible policy. The Canadian Forces stock contains such a class - the repairable or recoverable items. Although they make up six percent of the total stock, their dollar value is about 45 percent of the total investment at any given time. In order to satisfy demands, a spare stock of these items should be kept to guard against stock-outs, for resupply times are non-zero. Thus the optimal inventory policy (spare stock) must consider the tension between the high cost of this spare stock and the risk of stock out. This is, in broad terms, the nature of the problem treated by METRIC.

2. Before describing this situation any further it is imperative that clear definitions of terms be made here. The term <u>reorder</u> will be taken to mean a request for repair. An item is in <u>resupply</u> if it is in the process of being repaired. Let x be the number of units on order at a random point in time and let s be the total spare stock level. Then if $x < s$, these x units are said to be in <u>routine resupply</u>. If $x > s$, then $x - s$ units are said to be on <u>backorder</u>. Thus the total spare stock equals the stock on hand plus on order minus back-orders. As we shall see, a driving factor in establishing this spare stock level s is the average time required in resupply or <u>mean time to repair</u> (MTTR). In turn this resupply time is

directly affected by the geographical hierarchy of the repair facilities. This partitions the repair process into echelons. For example a unit removed from an equipment (aircraft for example) will be brought into the repair shop at the base out of which this equipment operates. If the unit cannot be repaired there, it will be sent to a DND depot. Again if repairs cannot be done there it will be sent to an outside contractor. Thus first echelon will be the base, second echelon will be an in-house depot and third echelon a contractor. Another driving factor defining the stock level s is the mean time between failures (MTBF), as this establishes the demand rates and thus the pressure on the spare stock.

3. METRIC, which is short for "Multi-Echelon Technique for Recoverable Item Control", was developed as a systems approach to initial provisioning of repairable items for a weapon system. In particular, the total investment in repairable spares and their distribution is related to the efficiency with which they support the major equipment of the weapon system. This permits life cycle costing and comparisons of different alternatives when purchasing a weapon system. In particular with estimates of failure rates and servicing rates under given economic or operational constraints, it will optimize a given measure of support function. Also, given existing stock levels at various bases and depots, it can evaluate the level of support provided by this allocation and can redistribute the stock to optimize the measure of support. METRIC optimizes only in terms of what stock is bought and/or how it is distributed. It does not optimize for instance against repair policy, scheduling or transportation systems between bases and depots. In what follows, the word equipment will refer to the major equipment, a certain number of which form the weapon system. An equipment is assumed to be made up of a certain number of items (type). A particular repairable piece of a certain item type will be referred to as a unit of that item.

Also in what follows bracketed numbers (e.g. (3)) in the text give the pertinent references. The symbol □ will refer to end of proofs.

THE METRIC MODEL

4. The METRIC model has a two echelon structure, that is there are a certain number of bases being supported by one central depot or by a series of depots each responsible for servicing certain bases or for repairing certain items only. When a unit comes into a base for repair it has a certain probability r of being repaired at the base and a probability 1-r of being repaired at a depot (see figure 1). The two basic parameters of the model determining the flow of repairables through the system are the MTBF and the MTTR. The model assumes that demands are governed by a compound Poisson distribution with a certain variance to mean ratio which is identical for all items (see II - Demand Process). The MTBF's provide the customer arrival rates for the compound Poisson distribution and a theorem of Palm tells us we can use the MTTR as the time period in the compound Poisson distribution in order to calculate the state probabilities (that x units are in resupply at a random point in time). The MTTR is a composite of the base repair time (BRT), the depot repair time (DRT) and the order and shipping time (OST). From these probabilities the measure of performance, availability, can be computed for each item. The availability of an item is simply the probability that a demand can be satisfied by the stock on hand, i.e. the probability that no backorders exist at a given point in time (see II - Measure of Supply Performance). As the failures of the items comprising an equipment are assumed to be independent, the product of the items' avilabilities gives the probability that no backorders exist on any item at a random point in time, i.e. the fraction of time all equipments are capable of operating.

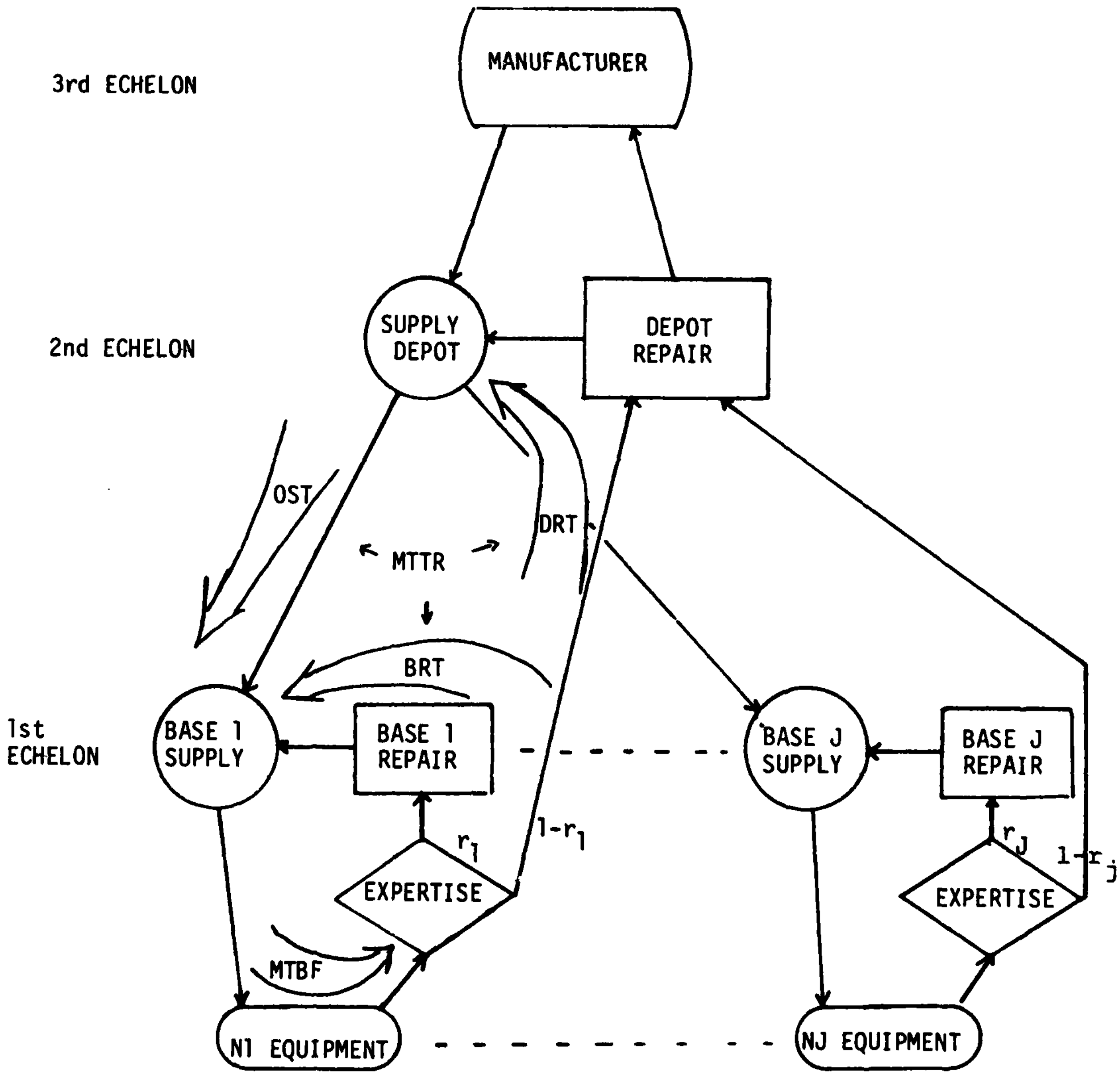


FIGURE 1 - FLOW OF REPAIRABLES

This is called the <u>system availability</u>. If there are n equipments operating, the availability of one equipment chosen at random can be obtained by taking the n^{th} root of the system availability.

5. At initial provisioning estimates of the MTBF's are at best uncertain. The METRIC model compensates for this by using a Gamma distribution for the true demand with the estimate of MTBF as its mean. This distribution is integrated with the availability function to obtain a mathematical expectation of the latter. If demand data are available, observations will be used to modify this gamma distribution using a Bayesian technique (see II - Demand Prediction).

6. The METRIC model uses marginal analysis to obtain the most efficient allocation of stock or investment. It can operate in three different modes. First, given an availability target for each equipment it will minimize the investment, or given an investment target it will maximize the availability for each equipment. Table 3 of (3) is reproduced here in Table I and gives two examples of METRIC calculations, one with an investment constraint and one with an availability constraint. In each it is assumed that two equipments are located at each of the four bases all with the same operational activity per year (see III). Second, it can redistribute an existing stock distribution so as to optimize the availability. Third, it can evaluate an existing stock, that is, it can determine the level of availability for a given stock distribution (see IV, paragraph 2).

TABLE I

EXAMPLES OF METRIC CALCULATIONS

MONEY CONSTRAINT OF $25,000

ITEM	BASE* REPAIR FRACTION	BASE REPAIR TIME (MON)	DEPOT REPAIR TIME (MON)	UNIT COST ($)	ESTIM. FAILURES PER YEAR (SYS)	METRIC'S STOCKAGE POLICY TOTAL	PIPELINE	BASES 1 2 3 4	ITEM AVAIL. %
1	100%	.50	0	100	9.63	8	0	2 2 2 2	.9964
2	50%	.50	2.0	400	4.89	6	2	1 1 1 1	.9877
3	20%	1.0	2.0	1000	7.74	7	3	1 1 1 1	.9728
4	0%	0	2.0	3000	3.00	5	1	1 1 1 1	.9805

Total Investment - $25,200

System Availability - 93.86

Equipment Availability - $(93.87)^{1/8}$ = 99.21%

EQUIPMENT AVAILABILITY CONSTRAINT OF 95%

ITEM	BASE* REPAIR FRACTION	BASE REPAIR TIME (MON)	DEPOT REPAIR TIME (MON)	UNIT COST ($)	ESTIM. FAILURES PER YEAR (SYS) †	METRIC'S STOCKAGE POLICY TOTAL	PIPELINE	BASES 1 2 3 4	ITEM AVAIL. %
1	100%	.50	0	100	9.63	4	0	1 1 1 1	.9647
2	50%	.50	2.0	400	4.89	5	1	1 1 1 1	.9792
3	20%	1.0	2.0	1000	7.74	4	0	1 1 1 1	.8244
4	0%	0	2.0	3000	3.00	1	1	0 0 0 0	.8170

Total Investment - $9,400

System Availability - 63.62%

Equipment Availability - $(63.62)^{1/8}$ = 95%

*(% of failed items which can be repaired at base level)

† The number of applications of each item in the equipment and the total annual operation hours are accounted for here.

N.B. It is assumed that the equipment and activity are equally distributed across the bases.

THE MODEL'S PARAMETERS

7. Some of the parameters have been mentioned already. We list here the primary parameters required for the model.

a. By Systems

- number of bases
- number of items
- number of equipments
- change in level of operation from data period to prediction period

b. By Item

- unit cost
- MTBF
- redundancy within an equipment
- observed demand
- number of months of observed demand

c. By Item and Base or Depot

- average base repair time (BRT)
- average depot repair time (DRT)
- fraction of failures that are base repairable
- average ordering and shipping time between a base and a depot (OST)
- number of equipments at each base
- minimum and/or maximum stock levels.

THE MODEL'S ASSUMPTIONS

8. Next we list the major structural assumptions of the METRIC model.

a. Demand Distribution: demand for each item is assumed to be a logarithmic Poisson process with a variance to mean ratio assumed constant across all items. Demand is also assumed stationary over the prediction period though it is possible to vary the level of operations from the data period.

b. Repair Decision: the decision to repair a unit at base level or depot is a function only of the type of malfunction and base maintenance capability. However, because of theoretical constraints it is assumed that a customer will have all his demands repaired at base level or all at the depot. Finally, at the depot it is assumed that service is done on a first-come first-served basis and that no batching of items for repair is permitted.

c. Lateral Resupply: lateral resupply between bases is not taken into account in the model. Resupply at a given base in the model comes from the base maintenance shop if repair is done at that base; otherwise it comes from the depot.

d. No Condemnations: the condemnation rate is assumed to be zero; that is, an item is repairable at base or depot regardless of its age or previous number of repairs.

e. All items comprising an equipment are considered equally important to the operation of the equipment and are independent from one another (probabilistically in their failures).

REFERENCES

9. The original METRIC model was developed by C. Sherbrooke (28). Shortly afterwards, a computer program was put together (8). A non-technical description can be found in (29). This model as developed by Sherbrooke used expected number of backorders as its performance measure. When the CF inherited the model, preference was indicated for availability as the measure of performance. Studies have been done (31), (20) indicating that the difference in stock allocations incurred by switching from the "backorder" to the "availability" measure is marginal. The changes were made (13) and the version resulting was called METRIC 2. During the period of implementation several reports were produced on applications to actual purchases ((6), (15), (16), (19)) and various studies were performed on the validity of the model ((3), (4), (5), (9)). Though it has been difficult to validate the model and though it is clear that there are shortcomings, there are at present no alternative means that take a systems approach to the problem of sparing in the acquisition of capital equipment. In response to these shortcomings, work has been done to get METRIC to behave more closely to the environment in the Canadian Forces Supply System (CFSS). Adaptation for more bases, more spares and more echelons were considered ((18), (22)). Modifications to include indent levels (or part hierarchies) have been studied as well ((21), (1)). At the moment METRIC is used exclusively for initial provisioning of spares. There is a need for METRIC's capability to follow sparing into the reprovisioning phase

of the CFSS, as spares bought initially on METRIC's allocation should be re-evaluated on the same basis. Some initial consideration has been given to this problem (2). Recently work outside DND has been done on problems of indent level, three echelons, finite repair capacity or repair facility breakdown ((24), (14), (7), (17), (23), (25)).

II - METRIC'S PROBABILITY MODEL

DEMAND PROCESS

10. Customer demand for each item is assumed to be a logarithmic Poisson process. This is a member of the compound Poisson family. These terms will now be defined. The following discussion refers to the demand process of one item at base level. Capital letters will be used to denote random variables and the corresponding lower case letters will represent values taken on by the random variable.

11. In a Poisson process the distribution of customer arrivals is given by the following probability function:

$$P(x|\lambda t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

This is the probability that x customers arrive in an arbitrary time interval of length t. This formula can be derived from the following two assumptions:

a. the probability of a single arrival in a small time Δt is $\lambda\Delta t$ (λ = "arrival rate");

b. the probability of more than one arrival in Δt is negligible.

Note that t is not absolute time but rather the length of an arbitrary interval of time. Here each customer arrives with one demand and hence the demand process is said to be a Poisson demand process. The mean of this distribution is λt and this value is also its variance. Finally the mean time between arrivals (MTBA) (or mean time between failures (MTBF) as failures generate arrivals) is $\frac{1}{\lambda}$. These are standard results which can be found in text books dealing with this subject (see (27; sec. 2-4) for instance).

12. In a compound Poisson process customers arrive as in a Poisson process at a rate λ with one or more demands given by a discrete positive random variable W with probability distribution function f(w) called the compounding distribution function. Thus in an interval of length t, the total number of demands is given by:

$$X(t) = \sum_{i=1}^{N(t)} W_i$$

where N(t) is the number of customers arriving in that interval, as given by the Poisson process and $\{W_i\}_{i=1,\ldots,N(t)}$ are independent identically distributed random variables of the demands of customers arriving in the interval. We would again like to describe the probability $P(x|\lambda t) = Pr(X(t)=x)$ of x demands occurring in an interval of length t when customers arrive at a rate λ. This time, since each customer must have at least one demand the probability of y customers arriving with x demands $(y \leq x)$ is

$$Pr(N(t)=y \text{ and } X(t)=x) = Pr(N(t)=y) \cdot Pr(X(t)=x|N(t)=y)$$

$$= \frac{(\lambda t)^y e^{-\lambda t}}{y!} \cdot Pr\left(\sum_{i=1}^{y} W_i = x\right)$$

In the last term we note that for a fixed y, the variable $\sum_{i=1}^{y} W_i$ has a distribution called the y-fold convolution of f (the distribution of W) with itself. The probability that the variable takes the value x is denoted $f^{y*}(x)$. Thus to form $P(x|\lambda t)$ we must add these probabilities over all y's producing x demands, i.e.

$$P(x|\lambda t) = \sum_{y=1}^{x} \frac{(\lambda t)^{y} e^{-\lambda t}}{y!} \cdot f^{y*}(x)$$

To see how this distribution varies from the Poisson we can calculate the mean m and variance σ^2 (for details see Annex A). They are:

$$m = \lambda t \; E(W)$$

$$\sigma^2 = \lambda t \; E(W^2)$$

where $E(W)$ and $E(W^2)$ are the first two moments of f given by

$$E(W) = \sum_{w=1}^{\infty} w \; f(w)$$

$$E(W^2) = \sum_{w=1}^{\infty} w^2 \; f(w)$$

As the difference between the Poisson and the compound Poisson is caused by f a better measure of the difference would be to take the variance to mean ratio q as it eliminates the parameter λt of the Poisson leaving us with the following:

$$q = \frac{\sigma^2}{m} = \frac{\sum_{w=1}^{\infty} w^2 \; f(w)}{\sum_{w=1}^{\infty} w \; f(w)} \geq 1$$

Note that we have $q = 1$ if $f(w) = 0$ for all $w > 2$, that is if the compound Poisson is Poisson.

13. We must now specify a distribution $f(w)$. The following function is used as it gives rise to a convenient computational form for $P(x|\lambda t)$ and when $q < 3$ remains faithful to previously accepted distributions (q will be shown to be the same as the q in paragraph 12 above):

$$f(w) = \begin{cases} 0 & w = 0 \\ \dfrac{1}{w \quad \ell n \; q} \left(\dfrac{p}{q}\right)^{w} & \begin{array}{l} w = 1,2,3, \ldots \\ q = p + 1 > 1 \end{array} \end{cases}$$

We specify also that

$$\lambda t = k \; \ell n \; q \qquad\qquad k > 0$$

With $f(w)$ defined as such, the compound Poisson process is called a Logarithmic Poisson Process. From the Taylor series expansion for ℓn we have:

$$\sum_{w=1}^{\infty} \frac{\left(\frac{p}{q}\right)^{w}}{w} = -\ell n \left(1 - \frac{p}{q}\right) = \ell n \; q$$

so that $\sum\limits_{w=0}^{\infty} f(w) = 1$ as required by a p.d.f. Also the second defining equation above gives rather simple expressions for the mean and the variance to mean ratio of the compound Poisson

$$m = \lambda t \; E(W) = k \; \ell n \; q \sum_{w=1}^{\infty} \frac{w}{w \quad \ell n \; q} \left(\frac{p}{q}\right)^{w}$$

$$= k \left[\left(\frac{1}{1 - \frac{p}{q}} \right) - 1 \right] = k \; (q - 1)$$

$$\frac{\sigma^2}{m} = \frac{E(W^2)}{E(W)} = \frac{\sum_{w=1}^{\infty} \frac{w^2}{w\ \ell n\ q}\left(\frac{p}{q}\right)^w}{\frac{q-1}{\ell n\ q}} = \frac{\sum_{w=1}^{\infty} w\left(\frac{p}{q}\right)^w}{p}$$

$$= \sum_{w=1}^{\infty} w\,\frac{1}{q}\left(\frac{p}{q}\right)^{w-1} = \frac{1}{1/q} = q$$

The second last equality results from the fact that the infinite sum is the expected value of a geometric distribution with probability of success $\frac{1}{q}$. Thus the parameter q defining f(w) is the variance to mean ratio of the final compound Poisson process. Through the use of characteristic functions it is shown in Annex A that $P(x|\lambda t)$ is a negative binomial distribution, namely if q and k are the parameters of the logarithmic Poisson

$$P(x|\lambda t) = \binom{-k}{x}\left(\frac{1}{q}\right)^k\left(-\frac{p}{q}\right)^x \qquad p = q-1, \quad k = \frac{\lambda t}{\ell n\ q}$$

$$= \binom{k+x-1}{x}\frac{(q-1)^x}{q^{k+x}}$$

where by convention $\binom{y}{x} = \frac{y(y-1)\ \ldots\ (y-x+1)}{x!}$ is extended to permit y to be any real number.

The calculations of $P(x|\lambda t)$ can be done recursively

$$P(x+1|\lambda t) = \frac{(x+k)}{(x+1)}\,\frac{p}{q}\,P(x|\lambda t),$$

where $P(0|\lambda t) = 1/q^k$

MEASURE OF SUPPLY PERFORMANCE

14. There are several different measures of supply performance that can be used. The original Rand version of METRIC uses expected number of backorders at a random point in time. METRIC 2 uses the concept of availability (see (31)) the probability of no backorders at a random point in time. These values are determined by the probability distribution of the random variable X, the number of units in resupply. The steady state probability distribution h(x) for X is given by a theorem due to Palm (see Annex B) which says that if demand is Poisson then X is also Poisson which depends on the mean of the resupply time distributions (mean time to repair - MTTR) and not on the resupply time distribution itself. More precisely:

Theorem (Palm) - Let the demand for an item be Poisson with rate λ and the resupply time be an arbitrary probability distribution $\psi(t)$ with mean T. Then the steady state probabilities of x units in resupply are given by the Poisson with rate λT, i.e.

$$h(x) = \frac{(\lambda T)^x e^{-\lambda T}}{x!}$$

Thus the distribution h(x) of the number of units in resupply depends only on the mean T of the distribution $\psi(t)$ of the resupply times and not, for instance, on the variance of $\psi(t)$.

15. To adapt this to our situation of compound Poisson demands, we must assume that a resupply time drawn from $\psi(t)$ is applicable to all demands placed by that customer. In this case then the steady state probability that y customers are in the system is given by

$$h(y) = \frac{(\lambda T)^y e^{-\lambda T}}{y!}$$

Now to gear this to demands, since the resupply process is independent of the customer order size, this must be multiplied by the probability that y customers place x demands, i.e. by $f^{y*}(x)$, and summed over all values of $y < x$. Thus the probability that x units are in resupply is

$$P(x|\lambda T) = \sum_{y=0}^{x} \frac{(\lambda T)^y e^{-\lambda T}}{y!} f^{y*}(x)$$

and we get that the number of units in resupply is also compound Poisson depending on T and not $\psi(t)$.

16. The availability is the probability that no backorders occur, i.e. $\Pr(X \leq s)$ where s is the total spare stock. Denoting this by A(s) we have

$$A(s) = \sum_{x=0}^{s} P(x|\lambda T)$$

If as in the Rand version we were concerned about the expected number of backorders as a measure the formula would be

$$B(s) = \sum_{x=s+1}^{\infty} (x-s) P(x|\lambda T)$$

DEMAND PREDICTION

17. In order to use the compound Poisson we must determine the true mean demand $\theta = \lambda L \bar{f}$ for a period L where $\bar{f}$ is the expected value of the compounding distribution. It is assumed to be stationary but otherwise is unknown. Because inaccuracies of point estimates for θ are magnified in computations this method is not acceptable. Furthermore, initial estimates of demand have a tendency to be larger than the true mean, usually by a factor of two (4). Because of this inability to establish a true mean demand, a Bayesian

Procedure is used and will now be described. This procedure requires a prior probability distribution for the values of the true mean demand which is assumed to be a gamma distribution over some unit period of time

$$g(\theta) = \frac{1}{w^v \Gamma(v)} \theta^{v-1} e^{-\theta/w} \qquad \begin{array}{l} 0 < \theta < \infty \\ v,w > 0 \end{array}$$

with mean $m = vw$ and variance $\sigma^2 = vw^2$. The initial estimate of θ is taken as this mean. The prior probability distribution will eventually be meshed together with observed data for each item. However, data for different items will be observed over different time periods. If we are now interested in the true mean distribution over a period r times as long, then we require the distribution of $X = r\theta$. The Jacobian of this transformation is $\frac{1}{r}$ so that the distribution of X is

$$G(X) = g\left(\frac{X}{r}\right) = \frac{1}{w^v \Gamma(v)} \left(\frac{X}{r}\right)^{v-1} e^{-X/rw} \frac{1}{r}$$

$$= \frac{1}{(rw)^v \Gamma(v)} X^{v-1} e^{-X/rw}$$

This is again conveniently a gamma distribution with a mean r times the mean for the unit period gamma distribution and variance r^2 times the variance for the unit period. Thus it suffices to establish the true mean demand distribution for some convenient time period.

18. Since there are two parameters to be estimated, two estimation equations will be needed. As mentioned earlier one is the mean

$$\bar{M}_i = vw$$

for each item i. In the absence of further data we can turn once again to the variance to mean ratio. From experience an estimate of this value across all items seems appropriate though the above mentioned adjustments for different time periods must be made. This ratio will be called the prior distribution ratio and the computer program will be run for several different values of the ratio for comparison of the different resulting stockage policies.

19. Once the prior distribution $g(\theta)$ for an item is identified we want to modify the distribution to take into account the fact that u demands for the item have been observed in time L. This is done using Bayes theorem to obtain the following posterior distribution $\phi(\theta|u)$ for θ

$$\phi(\theta|u) = \frac{P(u|\theta/\bar{f})\ g(\theta)}{\int_{\xi} P(u|\xi/\bar{f})\ g(\xi)\ d\xi}$$

Division by the mean number of demands per customer $\bar{f}$ changes the mean demand rate to the mean customer arrival rate as is required by the compound Poisson probability function $P(x|\lambda t)$. Computations of availability $A(s)$ (see Section IV) are done with a time period T, the average response time. Thus the mean arrival rate $\theta/\bar{f}$ calculated on a time period L must be re-scaled by $\frac{T}{L}$ for the formula of $A(s)$. Furthermore, if there is a change in the level of operations by a factor of a from the data period to the prediction period, this factor should also multiply the customer arrival rate. Thus

$$A(s|\theta) = \sum_{x=0}^{s} P(x|\ a.\ \frac{T}{L}\ .\ \frac{\theta}{\bar{f}})\ .$$

Finally the values of $A(s|\theta)$ are weighted using $\phi(\theta|u)$ to obtain

$$A^*(s|u) = \int_\theta \phi(\theta|u)\ A(s|\theta)\ d\theta \quad .$$

20. The above integrals are approximated using sums on five to ten points. Each subinterval is referred to as a Bayes state and in the n^{th} Bayes state an appropriate value θ_n is chosen. This leads to the following approximations

$$\phi(\theta_n|u) \simeq \frac{P(u|\theta_n/\bar{f})\ g(\theta_n)}{\sum_k P(u|\theta_k/\bar{f})\ g(\theta_k)}$$

and

$$A^*(s|u) \simeq \sum_n \phi(\theta_n|u) \sum_{x=o}^{s} P(x|a.\frac{T}{n}.\frac{\theta_n}{\bar{f}})$$

Note that ϕ and g are no longer probability density functions but the derived state probability obtained from suitable partitioning of the θ axis. In the absence of demand data $\phi(\theta_n|u)$ reduces to $g(\theta_n)$. The evaluation of $P(x|\lambda)$ requires the two parameters q and k. The value of q is chosen usually between 1 and 2. Experience indicates (4) that a value of 1.1 or 1.2 is appropriate in the CF. As the mean of $P(x|\lambda)$ is $k(q-1)$ then in each Bayes state division of θ_n by q-1 will yield the appropriate value of k for that Bayes state.

III - THE OPTIMIZATION TECHNIQUE

THE OBJECTIVE FUNCTION

21. Let c_i = unit cost of item i, i=1,2,---, I where I is the number of different items

a = minimum acceptable system availability target

s_{i0} = depot stock for item i

s_{ij} = base j stock for item i, j=1,2,---,J where J is the number of bases

$A(s_{i0}\ ,\ s_{ij})$ = availability of item i at base j when base stock is s_{ij} and depot stock is s_{i0}.

The objective is to solve the following

$$\min_{\{s_{ij}\}} \sum_{i=1}^{I} c_i \sum_{j=0}^{J} s_{ij}$$

subject to

$$\prod_{j=1}^{J} \prod_{i=1}^{I} A(s_{i0}\ ,\ s_{ij}) \geqslant a$$

or more conveniently

$$\sum_{j=1}^{J} \sum_{i=1}^{I} \log A(s_{i0}\ ,\ s_{ij}) \geqslant \log a$$

If budgetary constraints are of primary concern the problem will be formalized as

$$\max_{\{s_{ij}\}} \sum_{j=1}^{J} \sum_{i=1}^{I} \log A(s_{i0}, s_{ij})$$

subject to

$$\sum_{i=1}^{I} c_i \sum_{j=0}^{J} s_{ij} \leqslant T$$

for some investment target T.

22. The first problem will be considered here. When solving, only minor changes in computing give the solution of this second problem.

MARGINAL ANALYSIS

23. Consider a fixed item i. As a function of s_{ij}, $\log A(s_{i0}, s_{ij})$ is a concave function, where s_{i0} is held fixed (see Annex D). Thus for each depot stock level s_{i0} of a fixed item i one can obtain a maximal value of $\sum_{j=1}^{J} \log A(s_{i0}, s_{ij})$ for a given base stock $S = \sum_{j=1}^{J} s_{ij}$. In fact as we build up to S we add one unit at a time giving it to that base which yields the largest increase in availability. Concavity means that at any base marginal increases in availability for each new addition is continually decreasing. Thus if a unit is placed at a base which gives the largest increase in availability, all subsequent additions to any base will produce lower increases in availability. This process of allocating units to bases one at a time does so in such a way that the largest marginal increases in $\sum_{j=1}^{J} \log A(s_{i0}, s_{ij})$ are produced first and when S is

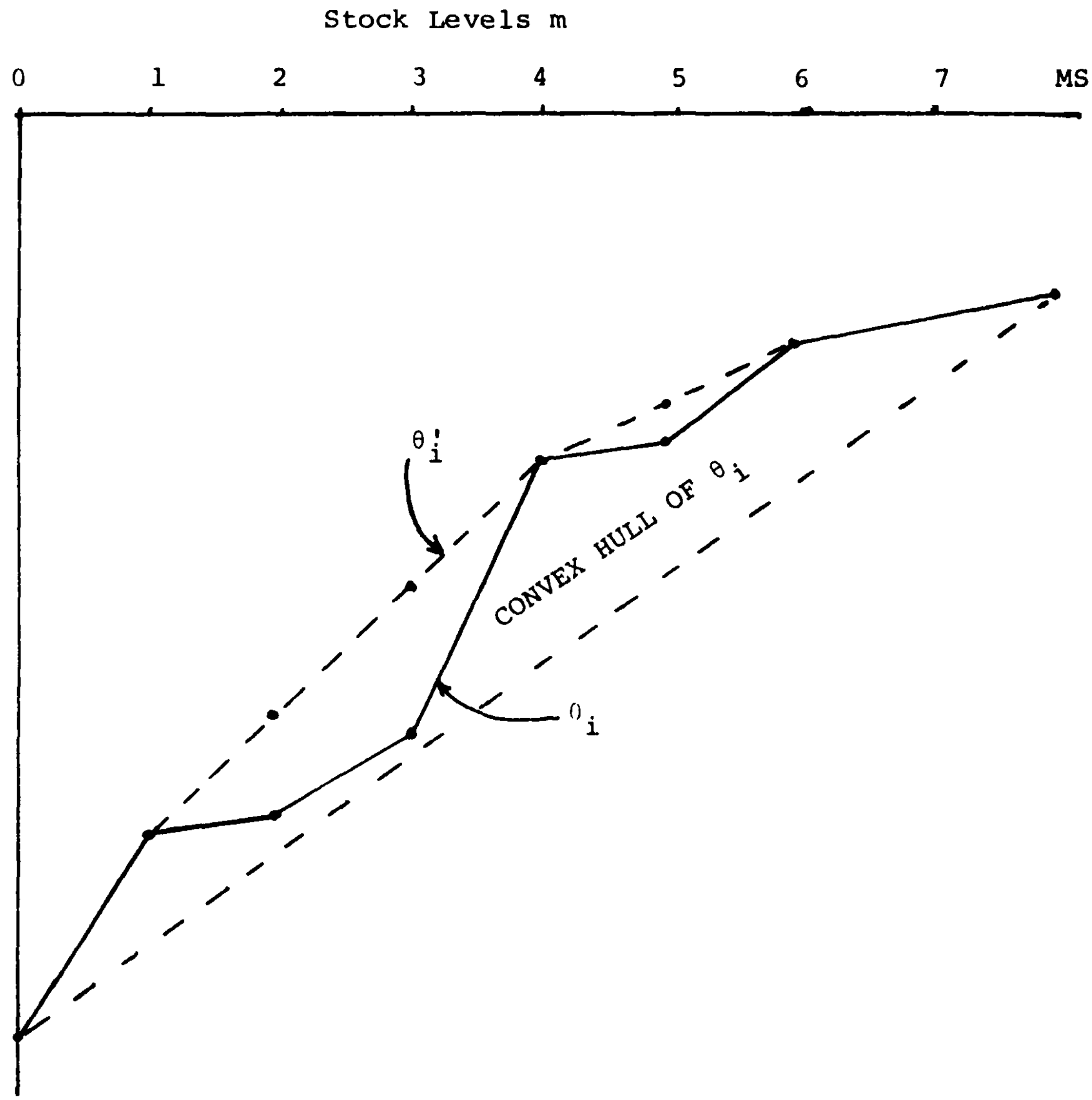
attained we will have the largest possible value for

$\sum_{j=1}^{J} \log A(s_{i0}, s_{ij})$ with a total base stock S and depot stock s_{i0}. Call this value $SA(s_{i0}, S)$. Now for each combination of base stock S and depot stock s_{i0} summing up to a total stock m ($m = s_{i0} + S$) pick that combination yielding the maximum value of $SA(s_{i0}, S)$ and call this value $\theta_i(m)$. This then yields the best possible distribution of m units of item i.

24. Now we must perform a multi-item distribution in the best way possible. If $\theta_i(m)$ were concave for each m then the same arguments as above would apply with the exception that costs must be accounted for. Thus as we build up the total stock of all items, at each step the next investment is allocated to the item that produces the largest increase in availability per dollar, that is whose value of

$$\frac{\theta_i(m+1) - \theta_i(m)}{c_i}$$

is the largest. In general $\theta_i(m)$ is not concave. It is, however, a montonically increasing function, as the availability resulting from the best allocation of m units plus an arbi. ·ary allocation of the $m + 1^{st}$ unit cannot be less than tl. ıvailability resulting from the best allocation for m unit. It is also bounded from above as $A(s_{i0}, s_{ij})$ being a probability, $\log A(s_{i0}, s_{ij}) \leqslant 0$ so that $\theta_i(m) \leqslant 0$. We therefore modify $\theta_i(m)$ by defining a new function $\theta_i'(m)$ so as to take values on the boundary of the convex hull of $\theta_i(m)$ (smallest convex set containing the graph of $\theta_i(m)$, see Figure 2) such that $\theta_i'(m) \geqslant \theta_i(m)$. The effect of this is to raise only those values at which $\theta_i(m)$ is not concave



MS = Maximum Stock Level

Graph of θ_i' obtained on upper edge of the Convex Hull (values raised at m = 2, 3, 5)

<u>Figure 2</u>

to obtain concavity. So now we would allocate the next investment to the item i with the largest value of

$$\frac{\theta_i'(m + 1) - \theta_i'(m)}{c_i}$$

Allocation terminates when the total availability just exceeds the target a.

25. Note that for a given item i, we don't terminate allocation at a level m such that $\theta_i(m) < \theta_i'(m)$, as the true availability of item i would be less than the value upon which a decision to take the m^{th} unit was made. In fact if $\theta_i(m) < \theta_i'(m)$ then the points on the graph of θ_i' at the values m-1, m and m+1 are collinear by construction of θ_i'. This means that

$$\theta_i'(m+1) - \theta_i'(m) = \theta_i'(m) - \theta_i'(m-1)$$

Hence

$$\frac{\theta_i'(m+1) - \theta_i'(m)}{c_i} = \frac{\theta_i'(m) - \theta_i'(m-1)}{c_i}$$

Thus the next best investment is again a unit of item i giving the same increase in θ_i' per unit dollar. Successive increments in item i are then continued until a value of m is reached such that $\theta_i'(m) = \theta_i(m)$ (which must occur as θ_i is bounded above). For example in Figure 2, if the second unit of item i was the last to be invested in then a third and a fourth are bought. At this point $\theta_i'(4) = \theta_i'(4)$. The point here is that investment decisions made on the artificial values of θ_i' is acceptable as they point to future large increases of θ_i which otherwise would be missed. Again in

Figure 2 the actual increase in θ_i going from 1 to 2 is so low that the comparison to increases incurred by other items for future investments may never permit purchase of the second unit and hence we may never benefit from the large increase obtained in going from 3 to 4 units of item i.

26. There is the potential problem of overshooting the target under this method but in practice this does not appear to be a serious problem. More serious is the fact that with a large number of items (> 300) the computations of the multi-item marginal analysis are extremely time consuming on a computer. The next section describes a computationally more efficient method.

THE GENERALIZED LAGRANGIAN METHOD

27. The general form of the original problem is the minimization of a cost function H(x) over a set S of strategies given constraints c_j, j=1,--, n on the production $c_j(x)$, j=1, ---, n of n products; i.e.

$$\min_{x \in S} H(x)$$

subject to

$$c_j(x) \geq c_j \qquad j=1,---,n$$

The following theorem by Everett (10) will simplify the problem by changing it to an non-constrained problem.

Theorem 1. Let τ_j, j=1,--,n be non-negative real numbers. If $x^* \varepsilon S$ minimizes the function $H(x) - \sum_{j=1}^{n} \tau_j c_j(x)$ over all $x \varepsilon S$, then x^* minimizes H(x) over the subset

$$S^* = \{x | c_j(x) \geq c_j(x^*) \text{for all}\} \subset S$$

Proof: Let x^* be as stated. Then

$$H(x^*) - \sum_{j=1}^{n} \tau_j\, c_j(x^*) \leqslant H(x) - \sum_{j=1}^{n} \tau_j\, c_j(x) \text{ for all } x \varepsilon S$$

$$H(x^*) \leqslant H(x) + \sum_{j=1}^{n} \tau_j\{c_j(x^*) - c_j(x)\} \quad \text{for all } x \varepsilon S$$

In particular this last inequality is true on $S^* \subset S$. However on S^*

$$\sum_j \tau_j \{c_j(x^*) - c_j(x)\} \leqslant 0$$

as $\tau_j > 0$ for all j. But this means that

$$H(x^*) \leqslant H(x) \text{ on } S^* \qquad \square$$

Notice that when solving the unconstrained problem we solve a constrained problem which is more restricted than the original problem and of course it is not known in advance. There is no guarantee that x^* exists or is unique for the unconstrained problem. If one such x^* does exist it gives the minimum cost possible without producing less (availability in our case) than x^* does. However, running through a spectrum of values for τ_j's gives answers under a series of different constraints and permits the values of $\{c_j(x^*)\}$ to approach those of $\{c_j\}$ through trial and error. Such evaluations under a series of different constraints is common procedure in operations research.

28. Theorem 1 permits us to reformulate the problem: for a given τ, find a stock strategy $\{s^*_{ij}\}$ optimizing

$$\min_{\{s_{ij}\}} \{ \sum_{i=1}^{I} (c_i \sum_{j=0}^{J} s_{ij})\} - \tau \sum_{j=1}^{I} \{ \sum_{i=1}^{I} \log A(s_{i0}, s_{ij})\}$$

Here we find the second important feature of theorem 1 for the form of our problem, namely the function to be minimized can be separated into independent functions, one for each item; i.e.

$$\min_{\{s_{ij}\}} \sum_{i=1}^{I} \{c_i \sum_{j=0}^{J} s_{ij} - \tau \sum_{j=1}^{J} \log A(s_{i0}, s_{ij})\}$$

and the minimum can be obtained by finding the minimum for each item i of

$$F_i = c_i \sum_{j=0}^{J} s_{ij} - \tau \sum_{j=1}^{J} \log A(s_{i0}, s_{ij})$$

Let τ be a fixed positive number, called the Lagrange multiplier, and i a fixed item. For each $m = \sum_{j=0}^{J} s_{ij}$,

$$F_i(m) = c_i \; m - \tau \; \theta_i(m)$$

gives the minimum value of F_i for that value m of total stock. The next theorem tells us that there is a unique minimum of F_i as a function of m and conditions under which to find it.

<u>Theorem 2</u> There exists a unique optimum $\bar{m}$ of

$$\min_m F_i(m)$$

satisfying

$$c_i - \tau \; \{\theta_i'(\bar{m}) - \theta_i'(\bar{m}-1)\} < 0 \leq c_i - \tau \; \{\theta_i'(\bar{m}+1) - \theta_i'(\bar{m})\}$$

where θ_i' is the concave extention of θ_i described earlier. For this $\bar{m}$, $\theta_i'(\bar{m}) = \theta_i(\bar{m})$.

The proof of this result is found in Annex C where $-\tau\theta_i$ is taken as θ a monotonically decreasing function bounded below by 0 (as $\tau > 0$ and θ_i is monotone increasing and bounded above by 0). Thus each item is treated individually (avoiding the multi-item approach) and its total stock $\bar{m}$ is determined by the conditions

$$\frac{\theta_i'(\bar{m}+1) - \theta_i'(\bar{m})}{c_i} \leqslant \frac{1}{\tau} < \frac{\theta_i'(\bar{m}) - \theta_i'(\bar{m}-1)}{c_i}$$

which are the above conditions restated. In words, we stop allocating stock of item i when the increase in availability per unit dollars drops below $\frac{1}{\tau}$. These conditions occur at an $\bar{m}$ where $\theta_i'(\bar{m})$ actually evaluates the total availability $\theta_i(\bar{m})$.

29. The major difficulty with the Lagrange method is that the τ is unknown. One could display several cost-effectiveness alternatives with different values τ or if an availability target is known, judicious trial and error runs can be carried out as described earlier.

IV - COMPUTATIONS

30. In this section we describe the computational steps required for the solution of the optimization problem described in section III. If the vector $(s^m_{i0}, s^m_{i1}, \ldots, s^m_{ij})$ is the best allocation of m units of item i as described in section III then the availability of item i was given by

$$\theta_i\ (m) = \sum_{j=1}^{J} \log A(s^m_{i0}, s^m_{ij})$$

The availability $A(s^m_{i0}, s^m_{ij})$ is determined by

$$A(s) = \sum_{x=o}^{s} P(x|\lambda T)$$

at each base j and for each item i as described in section II para. 17 where s is replaced by s^m_{ij} and T by $T_j\ (s_{i0})$, the mean resupply time at base j when the depot stock is s_{i0}. The formal steps are described next. Of course any expression in the following involving $P(x|\lambda t)$ is modified using the Bayesian procedure described in paragraph 19 section II, when observed demand data is available.

Step 1. *For each item i and depot stock level s_{i0} calculate the resupply time $T_j(s_{i0})$.* Since a fraction r_j of the demands are processed at base j, if the average base repair time is A_j then the contribution to resupply time at base j would be r_jA_j. If an item leaves base j to be repaired at the depot the time required would be the order and shipping time O_j if there was a infinite amount of depot spare stock, but would be O_j + D, where D is the average depot repair time (D actually consists of the average time for an item to be shipped from base to depot and to be repaired at depot), if that spare stock is zero. Thus the actual resupply time is

$$O_j + \delta(s_{i0})D$$

where the function $\delta(s_{i0})$ represents the ratio of expected number of units incurring delays during period D to the expected total demand during period D. As demands come in from all bases the customer arrival rate is

$$\lambda = \sum_{j=1}^{J} (1-r_j)\ \lambda_j\ a_j,$$

where a_j is the change in program factor from data period to prediction period at base j. This follows from Theorem 3 of Annex A and the fact that the customer arrival rate from base j is $(1-r_j)\ \lambda_j\ a_j$. Thus with a stock level of s_{i0} for item i at the depot, the expected number of delays in time D is the expected backorder (see para. 16)

$$B(s_{i0}|\lambda D) = \sum_{x=s_{i0}}^{\infty} (x-s_{i0})\ P(x|\lambda D)$$

and the expected total demand in time D is $\lambda D\bar{\bar{f}}$. Thus

$$\delta(s_{i0}) = \frac{B(s_{i0}|\lambda D)}{\lambda D\bar{\bar{f}}}$$

As the fraction of items going to the depot for repair is $1-r_j$ the contribution to the resupply time for these items is

$$(1-r_j)\quad (O_j + \delta(s_{i0})\ D)$$

Finally we compute

$$T_j(s_{i0}) = r_j\ A_j + (1-r_j)\quad (O_j + \delta(s_{i0})\ D)$$

<u>Step 2</u>. *For each depot stock level s_{i0} compute the availability for all stock levels s_{ij} at base j.* Thus we calculate for various levels of s_{ij}

$$A(s_{i0}\ ,\ s_{ij}) = A(s_{ij}|\lambda_j\ T_j\ (s_{i0})\ )$$

In this way we obtain the following table of availability for each i and value s_{i0}

bases \ s_{ij}	0	1	2	3	---	MBST
1						
2						
3						
4						
⋮						
J						

This table is calculated up to the maximum allowed level of stock MBST for item i at base j.

<u>Step 3</u>. *Compute* $SA(s_{i0}\ ,\ S) = \max\limits_{\sum_j s_{ij}=S} \sum_j \log A(s_{i0}\ ,\ s_{ij})$

With s_{i0} fixed, then according to section III paragraph 23, units of an item i can be distributed one at a time where the next unit is added to that base where the largest increase in availability will occur. Thus we can form the following table of best availability possible for a stock level S of item i when the depot has s_{i0} stock of item i. The calculations are done up to the maximum system stock level MSTAL allocatable to bases for item i and for each depot stock level up to its maximum MDST.

s_{io} \ S	0	1	2	3	---	MSTAL
0						
1						
2						
3						
⋮						
MDST						

<u>Step 4</u>. *Compute* $\theta_i(m) = \max\limits_{s_{i0}+S=m} SA(s_{i0}, S)$

For these computations we look along the diagonals of the table in Step 3 defined by

$$s_{i0} + S = \text{constant} = m$$

and choose the maximum value in the diagonal. This is the best allocation of m units of item i.

Step 5. *Marginal Analysis for the multi-item problem.* As $\theta_i(m)$ may not be concave the concave extension θ_i' of θ_i must be performed first. Next $\Delta\ \theta_i'(m) = \theta_i'(m) - \theta_i'(m-1)$ is calculated. At each step, the next investment is allocated to the item that produces the largest value of $\frac{\Delta\ \theta_i'(m)}{c_i}$ where c_i is the unit cost of item i. Allocation terminates when an investment target or a system availability target is just exceeded. If an optimal allocation of stock using the Lagrange multiplier method is required then an item by item allocation is performed, where allocation terminates for an item i when a total stock level m is reached such that

$$\frac{c_i}{\Delta\ \theta_i'(m)} < \tau \leqslant \frac{c_i}{\Delta\ \theta_i'(m+1)}$$

First estimates for τ may be obtained, for instance, by taking a small item sample, solving using the multi-item approach and if $\bar{m}$ is the optimal stockage policy for a sample item i use

$$\tau = \frac{c_i}{\Delta\theta_i'(\bar{m})}$$

or average this value over the sample, etc. (Note that budget or availability target values would have to be down graded proportionately to the sample size.)

31. There are several different ways to use this sequence of calculations. If we want an evaluation of the performance of existing stock and investment cost, computations can stop at step 2. If a redistribution of existing stock to improve availability is desired then steps 1 to 4 will optimally reallocate total assets among the bases and the depot. In the five step calculations minimum stock levels were taken as zero. If minimum stock levels of an item at base or system level are specified then computations for values lower than these levels will be omitted. After the above allocation is completed the total system investment and system availability are computed.

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THE COMPOUND POISSON PROCESS

Here we derive the mean and variance of the compound Poisson distribution and show that the logarithmic Poisson process has the probability states of the negative binomial distribution. To this end the concept of the characteristic function is useful in establishing relationships. If the random variable X has a probability distribution function g(x) then the characteristic $\psi_g(u)$ is defined by

$$\psi_g(u) = E(e^{iuX})$$

where u is a real number, $i = \sqrt{-1}$ and E is the expected value. We note that distinct probability distributions have distinct characteristic functions (see [12;XV.3, Theorem 1]).

Lemma 1. Let X and Y be two independently distributed random variables with probability distribution functions f and g respectively. Then

$$\psi_{f*g}(u) = \psi_f(u) \cdot \psi_g(u)$$

where f*g is the convolution of f and g (i.e. the p.d.f. of X + Y).

Proof: $\psi_{f*g}(u) = E(e^{iu(X+Y)}) = E(e^{iuX} \cdot e^{iuY})$

$$= E(e^{iuX}) \cdot E(e^{iuY}) = \psi_f(u) \cdot \psi_g(u)$$

□

<u>Theorem 1</u>. Let $P(x|\lambda t)$ be the p.d.f. of a compound Poisson distribution with demand p.d.f. $f(w)$(see para 12). Then

$$\psi_p(u) = e^{\lambda t\ (\psi_f(u)-1)}$$

Proof: $\psi_p(u) = E\ (e^{iu\ X(t)}) = \sum_{x=0}^{\infty} P(x|\lambda t)\ e^{iux}$

$$= \sum_{x=0}^{\infty} \left[\sum_{y=0}^{\infty} \frac{(\lambda t)^y}{y!}\ e^{-\lambda t}\ f^{y*}(x) \right] e^{iux}$$

$$= e^{-\lambda t} \sum_{y=0}^{\infty} \frac{(\lambda t)^y}{y!} \sum_{x=0}^{\infty} f^{y*}(x)\ e^{iux}$$

$$= e^{-\lambda t} \sum_{y=0}^{\infty} \frac{(\lambda t)^y}{y!}\ \psi_{f^{y*}}(u)$$

$$= e^{-\lambda t} \sum_{y=0}^{\infty} \frac{(\lambda t)^y}{y!}\ (\psi_f(u))^y \quad \text{by lemma 1}$$

$$= e^{-\lambda t}\ e^{\lambda t\ \psi_f(u)}$$

$$= e^{\lambda t\ (\psi_f(u)-1)}$$

□

Remark: The change in the order of summation in the proof is valid since for any fixed x the sum over the y's is a finite sum (see [26; theorem 8.3]).

<u>Theorem 2</u> For $P(x|\lambda t)$ we have

$$m = \lambda t \; E(W)$$

$$\sigma^2 = \lambda t \; E(W^2)$$

Proof From the definition of $\psi_P(u)$

$$\frac{d \; \psi_P(u)}{du}\Big|_{u=0} = \sum_{x=0}^{\infty} P(x|\lambda t) \; ix = i \; E(X) = im$$

$$\frac{d^2 \; \psi_P(u)}{du^2}\Big|_{u=0} = \sum_{x=0}^{\infty} P(x|\lambda t)(ix)^2 = - E(X^2) \quad .$$

On the other hand from the expression of $\psi_P(u)$ in the previous theorem

$$\frac{d \; \psi_P(u)}{du}\Big|_{u=0} = e^{\lambda t \; (\psi_f(o)-1)} \; \lambda t \; \frac{d\psi_f(u)}{du}\Big|_{u=0}$$

$$= \lambda t \; i \; E(W) \qquad \text{as } \psi_f(o) = 1$$

$$\frac{d^2\ \psi_p(u)}{du^2}\Big|_{u=0} = \left(\frac{\lambda t\ d\psi_f(u)}{du}\Big|_{u=0}\right)^2\ e^{\lambda t(\psi_f(o)-1)}$$

$$+\ e^{\lambda t\ (\psi_f(o)-1)}\ \lambda t\ \frac{d^2\psi_f(u)}{du^2}\Big|_{u=0}$$

$$= (\lambda t\ i\ E(W))^2 + \lambda t\ (-E(W^2))$$

$$= -\lambda t\ [\lambda t\ E(W)^2 + E(W^2)] \quad .$$

Therefore

$$im = \lambda t\ i\ E(W)$$

$$m = \lambda t\ E(W)$$

and $\quad E(X^2) = \lambda t\ [\ \lambda t\ E(W)^2 + E(W^2)]$

so that $\quad \sigma^2 = E(X^2) - m^2$

$$= \lambda t\ E(W^2) \quad .$$

□

At the depot compound Poisson demands are coming in from all bases. That the net process at the depot is again compound Poisson follows from the following theorem.

Theorem 3. A sum of N independent compound Poisson process with arrival rates λ_i and compounding functions f_i, i=1, ..., N is a compound Poisson process with arrival rate $\lambda = \sum_{i=1}^{N} \lambda_i$ and compounding function f such that

$$\psi_f(u) = \sum_{i=1}^{N} \frac{\lambda_i \psi_{f_i}(u)}{\lambda} .$$

Proof: Since this is true for N=1, by induction assume it is true for $N \leq k$. Consider the sum $X = \sum_{i=1}^{k+1} X_i$ of k+1 independent compound Poisson processes. By assumption then $X' = \sum_{i=1}^{k} X_i$ is a compound Poisson process with arrival rate $\lambda = \sum_{i=1}^{k} \lambda_i$ and compounding function f such that

$$\psi_f(u) = \sum_{i=1}^{k} \frac{\lambda_i \psi_{f_i}(u)}{\lambda} .$$

If P is the probability distribution for $X = X' + X_{k+1}$ then by lemma 1 and theorem 1

$$\begin{aligned}
\psi_P(u) &= e^{\lambda t(\psi_f(u)-1)} \cdot e^{\lambda_{k+1} t(\psi_{f_{k+1}}(u)-1)} \\
&= e^{t(\lambda\psi_f(u) - \lambda + \lambda_{k+1}\psi_{f_{k+1}} - \lambda_{k+1})} \\
&= e^{t(\sum_{i=1}^{k} \lambda_i \psi_{f_i} + \lambda_{k+1}\psi_{f_{k+1}} - \lambda - \lambda_{k+1})} \\
&= e^{(\sum_{i=1}^{k+1} \lambda_i)\, t\left[\left(\sum_{i=1}^{k+1} \lambda_i \psi_{f_i} \Big/ \sum_{i=1}^{k+1} \lambda_i\right) - 1\right]} .
\end{aligned}$$

As X is the sum of two compound Poisson processes X' and X_{k+1}, it is by assumption compound Poisson and by uniqueness of representation by characteristic functions, its arrival rate must be $\sum_{i=1}^{k+1} \lambda_i$ and the characteristic of the compounding distribution must be $\sum_{i=1}^{k+1} \lambda_i \psi_{f_i} \Big/ \sum_{i=1}^{k+1} \lambda_i$. This shows the statement is true for N = k + 1. $\square$

Finally it is shown that the probability distribution $P(x|\lambda t)$ of a logarithmic Poisson process is a negative binomial distribution. The negative binomial distribution gives the probability of the r^{th} success at the $(r+k)^{th}$ trial, namely if s is the probability of success and t = 1-s the probability of failure then

$$\Pr(r^{th} \text{ success at } r+k^{th} \text{ trial}) = \binom{r+k-1}{k} s^r t^k$$
$$= \binom{-r}{k} s^r (-t)^k$$

where by convention $\binom{x}{r} = \dfrac{x(x-1) \ \ldots \ (x-r+1)}{r!}$ is extended to permit x to be any real number. The name of the distribution comes from the fact that

$$(1-t)^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k} (-t)^k$$

Note that for r=1 this is the geometric distribution.

By uniqueness of representation through characteristic functions it suffices to show that the characteristic functions of these distributions are identical. Recalling that the compounding distribution of the logarithmic process is given by

$$f(w) = \begin{cases} 0 & w = 0 \\ \dfrac{1}{w \ \ell n \ q} \left(\dfrac{p}{q}\right)^w & w = 1,2,3,\ldots \\ & q = p+1 > 1 \end{cases}$$

then $$\psi_f(u) = \frac{1}{\ell n \ q} \sum_{w=1}^{\infty} \frac{\left(\frac{p}{q} \ e^{iu}\right)^w}{w}$$

$$= \frac{1}{\ell n \ q} \left[-\ell n\left(1 - \frac{p}{q} \ e^{iu}\right)\right]$$

$$= \frac{1}{\ell n \ q} \left[\ell n \ q - \ell n\left(q - pe^{iu}\right)\right]$$

$$= 1 - \frac{\ell n \ (q - pe^{iu})}{\ell n \ q}$$

so that

$$\psi_p(u) = e^{\lambda t} \ \ \psi_f(u) - 1$$

$$= e^{k(-\ell n(q - pe^{iu}))} \qquad (\lambda t = k \ \ell n \ q)$$

$$= \left(\frac{1}{q - pe^{iu}}\right)^k$$

On the other hand, we show that this is the characteristic function of the following negative binomial distribution

$$g(x) = \binom{-k}{x} \left(\frac{1}{q}\right)^k \left(-\frac{p}{q}\right)^x \qquad q = 1-p$$

Thus

$$\psi_g(u) = \sum_{x=0}^{\infty} \binom{-k}{x} \left(\frac{1}{q}\right)^k \left(-\frac{p}{q}\right)^x e^{iux}$$

$$= \left(\frac{1}{q}\right)^k \sum_{x=0}^{\infty} \binom{-k}{x} \left(-\frac{p}{q} e^{iu}\right)^x$$

$$= \left(\frac{1}{q}\right)^k \left(1 - \frac{p}{q} e^{iu}\right)^{-k}$$

$$= \left(\frac{1}{q - pe^{iu}}\right)^k$$

PROOF OF PALM'S THEOREM

Let $F(t) = \int_0^t \psi(\xi)d\xi$ be the distribution function for ψ.

Proposition 1. The probability that any particular order placed in the interval 0 to t has arrived by time t is $\frac{1}{t}\int_0^t F(\xi)d\xi$.

Proof: First we consider the probability that a demand occurs between ξ and $\xi + d\xi$ knowing that it occurred in $[0,t]$. This is given by

$$\frac{Pr(1 \text{ demand in } [\xi,\xi+d\xi]) \cdot Pr(0 \text{ demands in } [0,\xi]\cup[\xi+d\xi,t])}{Pr(1 \text{ demand in } [0,t])}$$

$$= \frac{\lambda d\xi e^{-\lambda d\xi} \cdot e^{-\lambda(t-d\xi)}}{\lambda t e^{-\lambda t}} = \frac{d\xi}{t}$$

Next the probability that a unit ordered at time ξ will arrive by time t is $F(t-\xi)$. Thus knowing that at least one demand occurred in $[0,t]$, the probability that it occurred between ξ and $d\xi$ and that the unit ordered arrived by time t is

$$\frac{F(t-\xi)}{t} d\xi$$

Consequently the probability that any particular order placed in the interval 0 to t will arrive by time t is

$$\frac{1}{t}\int_0^t F(t-\xi) \quad d\xi$$

An obvious change of variable yields the result of the proposition.

Proposition 2. The expected resupply time can be expressed as

$$T = \int_0^\infty (1-F(\xi))d\xi$$

Proof: By definition

$$T = \lim_{b\to\infty} \int_0^b \xi dF(\xi)$$

We use integration by parts to obtain

$$\int_0^b \xi dF(\xi) = bF(b) - \int_0^b F(\xi)d\xi$$

$$= b(F(b) - 1) + \int_0^b (1 - F(\xi))d\xi$$

But $0 \le b(1 - F(b)) = b\int_b^\infty dF(\xi) \le \int_b^\infty \xi dF(\xi)$

which tends to 0 as $b \to \infty$, giving the desired result. □

Proposition 3 The steady state probabilities of x units in resupply are given by the Poisson process with rate λT.

Proof: Suppose the system has been operating for time t. If during this time n demands were placed the number x of units in resupply at time t is

$$x = n - \text{arrivals}$$

Thus n - x would have arrived. Under these circumstances the probability of x units in resupply, can be given by the binomial probability of n - x successes (arrivals) in n trials

with $p = \frac{1}{t}\int_0^t F(\xi)d\xi$ as the probability of success (proposition 1)

and x failures with probability of failure $q=1-p=\frac{1}{t}\int_0^t (1-F(\xi))d\xi$.

Thus the probability of x units in resupply (failure) from n demands (trials) is $B(x:n,q)$.

As the probability of getting n demands is $P(n|\lambda t)$ in time t, the probability that x units in resupply at time t is

$$Q(x|t) = \sum_{n=x}^{\infty} P(n|\lambda t)\, B(x;n,q)$$

$$= \sum_{n=x}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \cdot \frac{n!}{x!\ (n-x)!} \quad q^x\ p^{n-x}$$

$$= \frac{(\lambda t)^x}{x!} q^x\ e^{-\lambda t} \sum_{n=x}^{\infty} \frac{(\lambda t)^{n-x}}{(n-x)!} \quad p^{n-x}$$

$$= \frac{(\lambda tq)^x}{x!} \quad e^{-\lambda t} \sum_{n=x}^{\infty} \frac{(\lambda tp)^{n-x}}{(n-x)!}$$

$$= \frac{(\lambda tq)^x}{x!} e^{-\lambda t}\ e^{\lambda tp}$$

$$= \frac{(\lambda tq)^x}{x!} e^{-\lambda tq}$$

$$= P(x|\lambda tq)$$

By proposition 2, $tq = \int_0^t (1-F(\xi))d\xi \to T$ as $t \longrightarrow \infty$ so that

$\lim_{t\to\infty} Q(x|t) = P(x|\lambda T)$ showing that the limiting state probabilities of x units in resupply are Poisson, depending only on the mean resupply time T.

ANNEX C

PROOF OF THEOREM 2 - SECTION III

Here we prove the following:

Theorem. Let θ be a real valued monotonically decreasing function defined on the set of non negative integers and bounded from below. Let θ' be defined on the non negative integers taking values on the boundary of the convex hull of θ such that $\theta'(m) \leq \theta(m)$. If $c > 0$ there exists a unique minimum $\bar{m}$ of

$$(1) \qquad \min_m \; cm + \theta(m)$$

satisfying

$$(2) \qquad c+\theta'(\bar{m}) - \theta'(\bar{m}-1) < 0 \leq c+\theta'(\bar{m}+1) - \theta'(\bar{m})$$

In this case

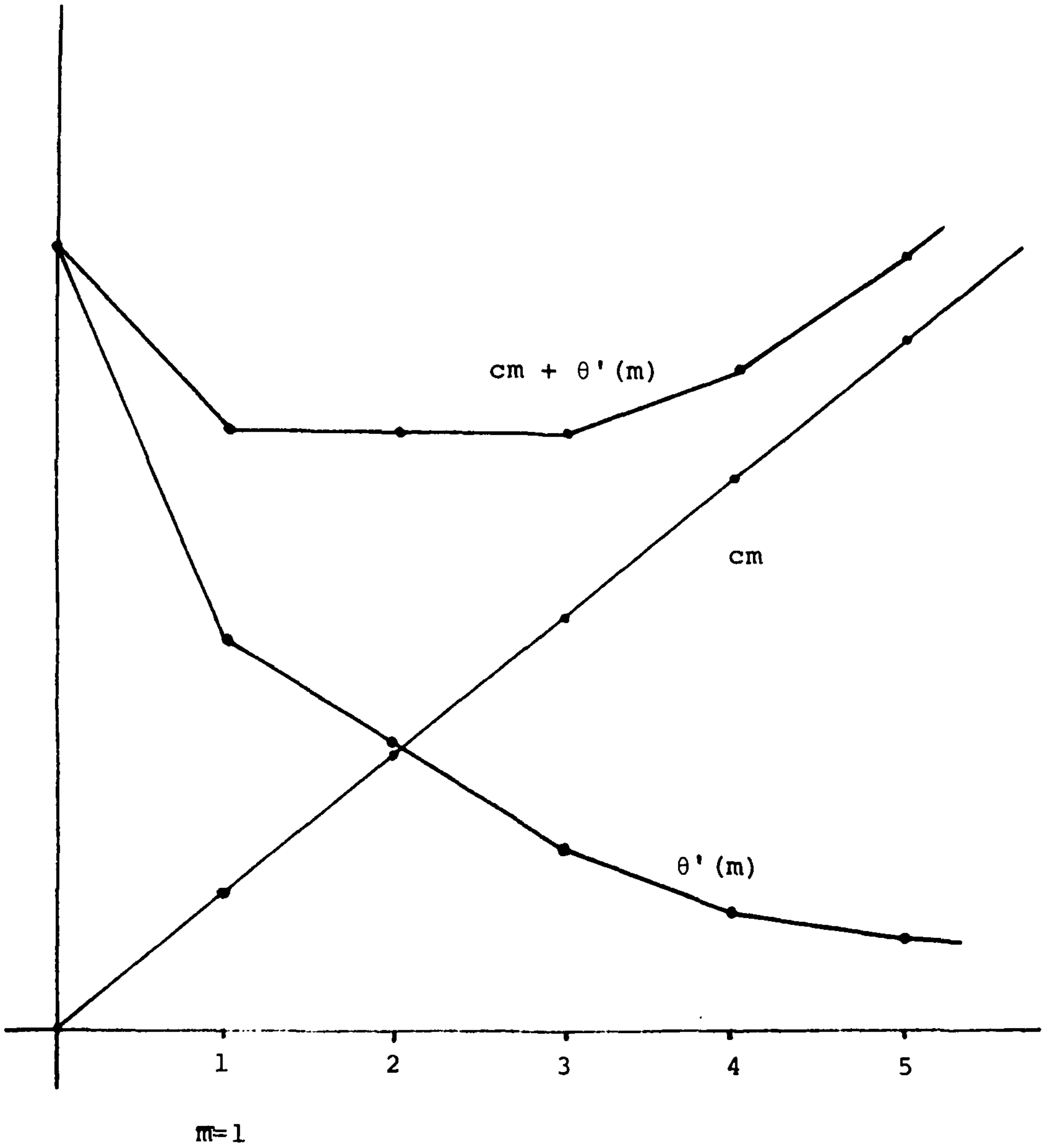
$$\theta'(\bar{m}) = \theta(\bar{m})$$

Proof: θ' is decreasing as any two consecutive points of the graph of θ' are on a line joining two points of the graph of θ which must have a non-positive slope. Also θ' is bounded from below. In fact as θ is bounded from below, by say b, then the graph of θ, $G(\theta)$, is contained in $B=\{(x,y)|y>b\}$ which is a convex set (see Figure 1). Thus $G(\theta') \subset$ {convex hull of $G(\theta)\} \subset B$ and so $\theta'(m)$ is bounded below by b as well. Finally θ' is clearly convex by definition. Thus $\theta'(m)$ converges and by convexity of θ', $\Delta(m) = \theta'(m) - \theta'(m+1)$ is monotonically decreasing. If we define $\theta'(-1) = \infty$ then $\Delta(-1) = \infty$. So by monotonicity of $\Delta(m)$ there is a unique m, call it $\bar{m} > 0$, such that

$$c < \Delta(\bar{m}-1)$$

and

$$c \geq \Delta(\bar{m})$$



m = 1, 2, 3 are all minima for cm + θ'(m)
but m̄ is chosen as the one with the smallest value

Figure 3

or

$$c - \Delta(\bar{m}-1) < 0 \leqslant c - \Delta(\bar{m})$$

which is the condition (2) stated in the theorem. Also as $cm+\theta'(m)$ is convex $\bar{m}$ is a a minimum of $cm+\theta'(m)$ as condition (2) can be written as

$$c(\bar{m}-1) + \theta'(\bar{m}-1) > c\bar{m} + \theta'(\bar{m})$$

$$c\bar{m} + \theta'(\bar{m}) < c(\bar{m}+1) + \theta'(\bar{m}+1)$$

(Heuristically, all minima of c m+ '(m) must be adjacent and condition (2) picks out the one with the smallest value - see Figure 2.) Next we show that $\theta(\bar{m}) = \theta'(\bar{m})$. If $\theta(\bar{m}) \neq \theta'(\bar{m})$ then by definition, the value of θ' at $\bar{m}$ has been lowered making the points of $G(\theta')$ at $\bar{m}-1$, $\bar{m}$ and $\bar{m}+1$ collinear. That is

$$\theta'(\bar{m}) - \theta'(\bar{m}-1) = \theta'(\bar{m}+1) - \theta'(\bar{m})$$

But this contradicts condition (2). Finally as

$$cm + \theta(\bar{m}) = c\bar{m} + \theta'(\bar{m}) < cm + \theta'(m) < cm + \theta(m) \quad \forall m$$

$cm + \theta(m)$ has a minimum at $\bar{m}$

PROOF OF THE CONCAVITY OF LOG A(s)

The availability A(s) used in METRIC 2 is defined by

$$A(s) = \sum_{x=0}^{s} P(x|\lambda t)$$

(see paragraph 16, section II), where $P(x|\lambda t)$ is the logarithmic Poisson distribution (para 13, section II) given recursively by

$$P(0|\lambda t) = 1/q^k, \quad P(x+1|\lambda t) = \frac{x+k}{x+1}\,\frac{q-1}{q}\,P(x|\lambda t) \quad x=1,2,\ldots$$

and where s is the stock level at a base. For brevity we will denote $P(x|\lambda t)$ simply by $P(x)$ and $\frac{x+k}{x+1}\,\frac{q-1}{q}$ by $C(x)$. In METRIC 2, -Log A(s) is used in the marginal analysis process instead of the Rand's expected backorder function. Convexity must be assured for use in the marginal analysis process. The expected backorder function is convex from its definition. This must be verified for - Log A(s). We will show here that Log A(s) is concave.

Before going on to the proofs, we note the following,

$$P(x+1) = C(x)\,P(x)$$

and $C(x) - C(x-1) = \frac{1-k}{(x+1)x}\,\frac{q-1}{q}$. Thus if $k>1$, $C(x)$ is a monotonically decreasing sequency converging to $\frac{q-1}{q} = 1-\frac{1}{q}$ where the variance to mean ratio q is greater than 1. If $k>1$, $C(x)$ is a monotonically increasing sequence converging to $1-\frac{1}{q}$ and if $k=1$ $C(x) = 1-\frac{1}{q}$ for all x.

The next two theorems give the proof of the concavity of Log A(s)

Theorem 1 $(C(s)-1)\ A(s) < P(s+1)$ for all $s > 0$.

Proof: We look at two cases.

1) If $k<1$ then as $C(s)$ increases to $1-\frac{1}{q}$ we have $C(s)<1-\frac{1}{q}$ so that $C(s)-1<-\frac{1}{q}<0$ for all s. Therefore, $(C(s)-1)\ A(s) < 0 < P(s+1)$ for all s.

2) If $k>1$ then we do a proof by induction on s. For $s=0$

$$\begin{aligned}(C(0)-1)A(0) &= (C(0)-1)\ P(0)\\ &= C(0)\ P(0) - P(0)\\ &= P(1) - P(0) < P(1)\end{aligned}$$

and the hypothesis is verified at s=o. Assume it is true for s-1, i.e.

$$(C(s-1)-1)\ A(s-1) < P(s)$$

Since C(s) is a decreasing sequence then $C(s) < C(s-1)$. The above inequality becomes

$$(C(s)-1)\ A(s-1) < P(s)$$
$$P(s+1) - P(s) + (C(s)-1)\ A(s-1) < P(s+1)$$
$$(C(s)-1)\ P(s) + (C(s)-1)\ A(s-1) < P(s+1)$$
$$(C(s)-1)\ (P(s) + A(s-1)) < P(s+1)$$
$$(C(s)-1)\ A(s) < P(s+1)$$

which is the hypothesis for s □

Theorem 2 Log A(s) is concave.

Proof: To show that Log A(s) is concave we must show $\Delta^2 \text{Log } A(s) < 0$, i.e. $\text{Log } A(s+2)-2 \text{ Log } A(s+1)+\text{Log } A(s)<0$
From theorem 1 at s+1 we have
$(C(s+1)-1)\ A(s+1) < P(s+2)$

Multiplying both sides by P(s+1) we obtain

$$(P(s+1)\ C(s+1) - P(s+1))\ \ A(s+1) < P(s+1)\ P(s+2)$$

$$(P(s+2) - P(s+1))\ \ A(s+1) - P(s+1)\ P(s+2) < 0$$

$$(A(s+1))^2 + (P(s+2)-P(s+1))\ A(s+1)-P(s+1)P(s+2)<(A(s+1))^2$$

$$(A(s+1)+P(s+2))\ \ (A(s+1)-P(s+1))<(A(s+1)^2$$

$$A(s+2)\ A(s) < (A(s+1))^2$$

$$\text{Log } A(s+2) + \text{Log } A(s) < 2 \text{ Log } A(s+1)$$

$$\text{Log } A(s+2) - 2 \text{ Log } A(s+1) + \text{Log } A(s) < 0$$

which is the required result. $\square$

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)

1. ORIGINATING ACTIVITY Department of National Defence Operational Research and Analysis Establishment	2a. DOCUMENT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP III

3. DOCUMENT TITLE

METRIC 2: The Mathematical Theory of a Sparing Model for Repairables

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (Last name, first name, middle initial)

VINCENT, P.A.

6. DOCUMENT DATE March 1980	7a. TOTAL NO. OF PAGES 59	7b. NO. OF REFS 31
8a. PROJECT OR GRANT NO. 96516-1	9a. ORIGINATOR'S DOCUMENT NUMBER(S) ORAE Memorandum No. M102	
8b. CONTRACT NO.	9b. OTHER DOCUMENT NO.(S) (Any other numbers that may be assigned this document)	

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This report describes METRIC 2, a mathematical sparing model for repairables. After discussing its general structure, the probability and optimization techniques are explored. Finally the sequence of steps required to solve a given problem is discussed. These are basically the steps used in the FORTRAN program of the model currently used in initial provisioning for capital equipment in accordance with the procedures of CF Logistics Management Instruction 1630.

Key Words

Optimal sparing model
Availability/budget
Inventory control
Recoverable item

DSIS
78-041

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